Gaussian estimates for Schrödinger perturbations *

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Abstract

We prove an optimal 4G Theorem for the Gaussian kernel. We also propose a new general method of estimating Schrödinger perturbations of transition densities, and give applications to the Gaussian kernel.

1 Introduction

A Schrödinger perturbation is an addition of an operator of multiplication to a given operator. On the level of inverse operators, the addition results in resolvent or Duhamel's or perturbation formula, and under certain conditions it yields von Neumann or perturbation series for the inverse of the perturbation. The subject is very wide, and we intend to study it in the case when the inverse operator is an evolution semigroup, in fact, a transition density. In this case a convenient and simple setup is that of integral operators on space-time, and the perturbation series has an exponential flavor due to repeated integrations on time simplexes. We propose a general method for pointwise estimates of the series, and we show its versatility by estimating transition densities of Schrödinger perturbations of heat kernels on $\mathbb{R}^d$.

In an earlier work, Bogdan, Jakubowski and Sydor [6] developed a technique for *sharp* pointwise estimates of Schrödinger perturbations $\tilde{p}$ of transition densities p and more general integral kernels on state space X by functions $q \geq 0$. The method rests on the assumption

$$\int_s^t \int_X p(s,x,u,z)q(u,z)p(u,z,t,y)\,dz\,du \leq [\eta + Q(s,t)]p(s,x,t,y), \qquad (1)$$

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where $s < t$, $x, y \in X$, $\eta < \infty$, and $0 \le Q(s,u) + Q(u,t) \le Q(s,t)$. The left-hand side of (1) defines the term $p_1(s,x,t,y)$ in the perturbation series,

$$\tilde{p}(s,x,t,y) = \sum_{n=0}^{\infty} p_n(s,x,t,y) \tag{2}$$

(see Section 2), and so $p_1(s,x,t,y)/p(s,x,t,y) \le \eta + Q(s,t)$. The bound is uniform in space and locally uniform in time, and it propagates as follows,

$$\begin{aligned} p_n(s,x,t,y) &\le p_{n-1}(s,x,t,y)\left[\eta + \frac{Q(s,t)}{n}\right] \\ &\le p(s,x,t,y)\prod_{k=1}^{n}\left[\eta + \frac{Q(s,t)}{k}\right]. \end{aligned} \tag{3}$$

Furthermore, if $0 < \eta < 1$, then (2) and (3) yield,

$$\tilde{p}(s,x,t,y) \le p(s,x,t,y)\left(\frac{1}{1-\eta}\right)^{1+Q(s,t)/\eta}, \tag{4}$$

and if $\eta = 0$, then

$$\tilde{p}(s,x,t,y) \le p(s,x,t,y)e^{Q(s,t)}. \tag{5}$$

The above estimates are sharp i.e. the ratio of upper bound of $\tilde{p}$ and (the trivial lower bound) p is bounded locally in time. In fact, as shown by Bogdan, Hansen and Jakubowski [4, Example 4.3 and 4.5] the exponential factors in (4) or (5) are nearly optimal. The estimates also apply to very general integral kernels on space-time, without assuming Chapman-Kolmogorov equations. It is now crucial to verify (1) for given p and q. To this end we usually try to split and estimate the singularities of the integrand $(u,z) \mapsto p(s,x,u,z)q(u,z)p(u,z,t,y)$ in (1). This is straightforward if q satisfies a suitable Kato-type condition and 3G Theorem holds for p (see [6, Remark 2] and the discussion in Section 2 and Section 4 below). The latter is the case, e.g., for the transition density of the fractional Laplacian as described by Bogdan, Hansen and Jakubowski [3, Corollary 11], [4, Example 4.13], and for the potential kernel of the stable subordinator [6, Example 2], because the functions have power-type asymptotics.

However, 3G fails for the Gaussian transition density due to its exponential decay. Motivated by this obstacle, the results of Zhang [26, 27] and the arguments of Jakubowski and Szczypkowski [17, 18], in Section 2 we propose a more flexible method of estimating Schrödinger perturbations of *transition densities*. The method employs an auxiliary transition density p^*, as an approximate *majorant* of p substituting for $p(\cdot,\cdot,t,y)$ in (1). In the second part of the paper we test our method on Gaussian-type estimates. Namely, in Section 3 we elaborate [26, inequality (4.4)] by giving a sharp estimate involving four Gaussian kernels (hence 4G), one of which may be interpreted

as the majorant. Then, in Section 4 we obtain precise (but necessarily not sharp) Gaussian estimates for the fundamental solution of Schrödinger perturbations of second order parabolic differential operators, recovering and improving existing results, which we discuss there at some length. We also describe a connection to second order differential operators. In Section 5 we give miscellaneous methodological comments.

Our inspiration mainly comes from [18] and [6]. Our ideas are also similar to those developed for Gaussian estimates in [26]. In particular, the condition (16) for our main Theorem 1 may be considered as a generalization of the Main Lemma 4.1 of [26], while the 4G inequality in Theorem 2 yields an alternative, synthetic justification of that lemma. Furthermore, the proof of [26, inequality (4.4)], yields (24), except for the optimal constant M. It is thus of interest that the approach of [26], which was taylor-made for the Gaussian kernel, has a more general context given by Theorem 1. Noteworthy, our approach is not restricted to Gaussian-type kernels. Further applications, e.g. to perturbations of the transition density of the 1/2-stable subordinator, will be given in a forthcoming paper.

2 General estimates of transition densities

Let X be an arbitrary set with a σ-algebra $\mathcal{M}$ and a (non-negative) σ-finite measure m defined on $\mathcal{M}$. To simplify the notation we will write dz for $m(dz)$ in what follows. We also consider the Borel subsets $\mathcal{B}$ of $\mathbb{R}$, and the Lebesgue measure, du, defined on $\mathbb{R}$. The *space-time*, $\mathbb{R}\times X$, will be equipped with the σ-algebra $\mathcal{B}\times\mathcal{M}$ and the product measure $du\,dz = du\,m(dz)$.

We will consider a *measurable transition density* p on space-time, i.e. we assume that $p:\mathbb{R}\times X\times\mathbb{R}\times X\to[0,\infty]$ is $\mathcal{B}\times\mathcal{M}\times\mathcal{B}\times\mathcal{M}$-measurable and the Chapman-Kolmogorov equations hold for all $x,y\in X$ and $s<u<t$:

$$\int_X p(s,x,u,z)p(u,z,t,y)\,dz = p(s,x,t,y)\,. \tag{6}$$

Let $q\colon \mathbb{R}\times X\to[0,\infty]$ be (nonnegative and) $\mathcal{B}\times\mathcal{M}$-measurable. The Schrödinger perturbation $\tilde{p}$ of p by q is defined by the series (2), where $p_0(s,x,t,y)=p(s,x,t,y)$,

$$p_1(s,x,t,y) = \int_s^t\int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu,$$

and for $n = 2, 3, \ldots$,

$$p_n(s,x,t,y) = \int\limits_s^t \int\limits_{u_1}^t \cdots \int\limits_{u_{n-1}}^t \int\limits_{(\mathbb{R}^d)^n} p(s,x,u_1,z_1)q(u_1,z_1)$$
$$p(u_1,z_1,u_2,z_2)\cdots q(u_n,z_n)p(u_n,z_n,t,y)dz_n\cdots dz_1 du_n \cdots du_1. \tag{7}$$

By Fubini-Tonelli, for $n = 1, 2, \ldots$, we have

$$p_n(s,x,t,y) = \int\limits_s^t \int\limits_X p(s,x,u,z)q(u,z)p_{n-1}(u,z,t,y)dzdu\,, \tag{8}$$

and

$$p_n(s,x,t,y) = \int\limits_s^t \int\limits_X p_{n-1}(s,x,u,z)q(u,z)p(u,z,t,y)dzdu\,. \tag{9}$$

By [3, Lemma 1], for all $s < u < t$, $x, y \in X$ and $n \in \mathbb{N}_0 = \{0, 1, \ldots\}$,

$$\sum_{m=0}^{n} \int\limits_X p_m(s,x,u,z)p_{n-m}(u,z,t,y)\,dz = p_n(s,x,t,y)\,. \tag{10}$$

By [3, Lemma 2], Chapman-Kolmogorov equations hold for $\tilde{p}$. Clearly, $\tilde{p} \geq p$.

Remark 1. The perturbation, say p_V, is given by the same formulae if $V : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ (takes on complex values), provided $p_{|V|}$ is finite. Then p_V converges absolutely and

$$|p_V| \leq p_{|V|}. \tag{11}$$

A detailed discussion of signed real-valued perturbations of transition densities is given in [3], with a positive lower bound for p_V which follows from Jensen's inequality. A probabilistic interpretation of p_n and p_V may also be found in [3].

Below we focus on upper bounds of $\tilde{p} = p_q$ for *transition densities* p and $q \geq 0$. This is a less general setting than that of [6], but within this setting our bounds, (17) and (18) below, hold under more flexible condition (16) on p and q. Namely, we consider another measurable transition density p^* and $C \geq 1$ such that for all $x, y \in X$ and $s < t$,

$$p(s,x,t,y) \leq Cp^*(s,x,t,y)\,. \tag{12}$$

We can estimate the cumulative effect of the integrations involved in (8). The following result is an analogue of [18, Lemma 5] and [4, Example 4.5].

Lemma 2. *Let $\theta \geq 0$ and $s_0 < \ldots < s_k = t$ be such that*

$$\int_s^{s_{i+1}} \int_X p(s,x,u,z)q(u,z)p^*(u,z,s_{i+1},y)\,dzdu \leq \theta\, p^*(s,x,s_{i+1},y), \tag{13}$$

for all $i = 0, \ldots, k-1$, $s \in [s_i, s_{i+1}]$ and $x, y \in X$. Then for every $n \in \mathbb{N}_0$,

$$p_n(s,x,t,y) \leq \binom{n+k-1}{k-1} \theta^n C^k p^*(s,x,t,y), \quad s \in [s_0,s_1],\ x,y \in X. \tag{14}$$

Proof. For $k = 1$, the estimate holds for $n = 0$ by (12), and then it holds for all $n \geq 1$ by induction, (8) and (13):

$$\begin{aligned}
p_n(s,x,t,y) &= \int_s^t \int_X p(s,x,u,z)q(u,z)p_{n-1}(u,z,t,y)dzdu \\
&\leq \theta^{n-1} C \int_s^t \int_X p(s,x,u,z)q(u,z)p^*(u,z,t,y)dzdu \\
&\leq \theta^n C p^*(s,x,t,y), \qquad \text{where} \qquad s \in [s_0,s_1], \quad x,y \in X.
\end{aligned}$$

If $k \geq 2$, then by (10), induction and Chapman-Kolmogorov for p^*,

$$\begin{aligned}
&p_n(s,x,t,y) = \sum_{m=0}^{n} \int_X p_m(s,x,s_{k-1},z)p_{n-m}(s_{k-1},z,t,y)\,dz \\
&\leq \sum_{m=0}^{n} \int_X \binom{m+k-2}{k-2} \theta^m C^{k-1} p^*(s,x,s_{k-1},z)\theta^{n-m} C p^*(s_{k-1},z,t,y)\,dz \\
&= \binom{n+k-1}{k-1} \theta^n C^k p^*(s,x,t,y)\,, \qquad \text{if } s \in [s_0,s_1],\ x,y \in X, n \in \mathbb{N}_0.
\end{aligned}$$

□

In passing we note that the assumption and conclusion in the statement of [18, Lemma 5] need a slight strengthening for the induction to work properly: each t_{i+1} in the assumption there should be replaced by τ in $[t_i, t_{i+1}]$, and each t in the conclusion should be replaced by τ in $[t_i, t_{i+1}]$ (then one proceeds as in the proof of Lemma 2 above). The correction does not influence applications of Lemma 5 or other results in [18].

We further let $Q \colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be *regular superadditive*, meaning that

$$Q(s,u) + Q(u,t) \leq Q(s,t)\,, \qquad \text{if} \quad s < u < t\,, \tag{15}$$

$Q(s,t) = 0$ if $s \geq t$, $s \mapsto Q(s,t)$ is right-continuous and $t \mapsto Q(s,t)$ is left-continuous (the continuity assumptions are rather innocuous, as we explain

later on in Lemma 12 and Lemma 13). We see that $t \mapsto Q(s,t)$ is non-decreasing and $s \mapsto Q(s,t)$ is non-increasing. For instance, if μ is a Radon measure on $\mathbb{R}$, then $Q(s,t) = \mu(\{u \in \mathbb{R} : s < u < t\})$ is regular superadditive. A regular superadditive Q is infinitely decomposable in the following sense.

Lemma 3. *Let $s \le t$, $k \in \mathbb{N}$ and $\theta \ge 0$ be such that $Q(s,t) \le k\theta$. Then $s = s_0 \le s_1 \le \ldots \le s_k = t$ exist such that $Q(s_{i-1}, s_i) \le \theta$ for $i = 1, \ldots, k$.*

Proof. We may and do assume that $k > 1$ and $(k-1)\theta < Q(s,t) \le k\theta$. Let $s_i = \inf\{u : Q(s,u) \ge i\theta\}$ for $i = 1, \ldots, k-1$. If $s \le u < s_i$, then $Q(s,u) < i\theta$, and so $Q(s, s_i) \le i\theta$. If $s_i < u < s_{i+1}$, then $Q(s,u) \ge i\theta$ and $Q(s,u) + Q(u, s_{i+1}) \le Q(s, s_{i+1}) \le (i+1)\theta$, thus $Q(u, s_{i+1}) \le \theta$. Letting $u \to s_i$ we obtain $Q(s_i, s_{i+1}) \le \theta$, which is also true if $s_i = s_{i+1}$. $\square$

Definition 4. *We write $q \in \mathcal{N}(p, p^*, C, \eta, Q)$ if $q \ge 0$ is measurable on space-time, p and p^* are measurable transition densities, $C \ge 1$, $\eta \ge 0$, Q is regular superadditive, and (12) and the following inequality hold for all $s < t$ and $x, y \in X$,*

$$\int_s^t \int_X p(s,x,u,z) q(u,z) p^*(u,z,t,y)\, dz\, du \le \big[\eta + Q(s,t)\big] p^*(s,x,t,y)\,. \quad (16)$$

The terms η and $Q(s,t)$ of (16) propagate differently in estimates of p_n below. We may think about η as giving a bound for instantaneous growth of mass, while Q gives a cap for growth accumulated over time (see [6] and [4] for such insight).

Theorem 1. *If $q \in \mathcal{N}(p, p^*, C, \eta, Q)$, then for all $s < t$ and $x, y \in X$,*

$$\tilde{p}(s,x,t,y) \le p^*(s,x,t,y) \left(\frac{C}{1 - 2\eta} \right)^{1 + \frac{Q(s,t)}{\eta}}, \qquad \text{if} \quad 0 < \eta < 1/2, \quad (17)$$

and

$$\tilde{p}(s,x,t,y) \le p^*(s,x,t,y)\, (2C)^{1 + 2Q(s,t)}, \qquad \text{if} \quad \eta = 0. \quad (18)$$

Proof. Let $k \in \mathbb{N}$. By Lemma 3, $s_0 = s < s_1 < \ldots < s_k = t$ exist such that $Q(s_{i-1}, s_i) \le Q(s,t)/k$ for $i = 1, \ldots, k$. By Lemma 2 with $\theta = \eta + Q(s,t)/k$, and by Taylor's expansion, for all $x, y \in X$ we get

$$\begin{aligned}
\tilde{p}(s,x,t,y) &\le \sum_{n=0}^{\infty} |p_n(s,x,t,y)| \\
&\le \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} C^k \left[\eta + Q(s,t)/k\right]^n p^*(s,x,t,y) \\
&= \left(\frac{C}{1 - \eta - Q(s,t)/k} \right)^k p^*(s,x,t,y),
\end{aligned}$$

For $\eta > 0$, we choose $k \in \mathbb{N}$ such that $(k-1)\eta \le Q(s,t) < k\eta$. For $\eta = 0$, we choose k satisfying $(k-1)/2 \le Q(s,t) < k/2$. This ends the proof. $\square$

Analogous results hold if we state our assumptions and conclusions for s, t in a finite time horizon: $-\infty < t_1 \le s < t \le t_2 < \infty$. If Q in Theorem 1 is bounded, then $\tilde{p} \le const.\, p^*$ uniformly in time. We may consider $q \in \mathcal{N}(p, p^*, C, \eta, Q)$ with $\eta < 1/2$, $q_1(u,z) = q(u,z)\,\mathbf{1}_{[0,1]}(u)$, and bounded superadditive function $Q_1(s,t) = Q((s \vee 0) \wedge 1, (t \wedge 1) \vee 0)$. Then,

$$\int_s^t \int_X p(s,x,u,z) q_1(u,z) p^*(u,z,t,y)\, dz du \le \big[\eta + Q_1(s,t)\big] p^*(s,x,t,y)\,.$$

Thus, by Theorem 1, $\tilde{p} \le c p^*$ uniformly in time. If p^* is not comparable with p in space then the estimates in Theorem 1 cannot be sharp. This is regrettable, but quite common, e.g., in Schrödinger perturbations of Gaussian kernel discussed later on in the paper. The role of p^* is similar to that of f in [4, Theorem 3.2], but the results of [4] do not apply in the present setting, if $p \ne p^*$. If (16) holds with p^* replaced by p, then we may take $p^* = p$ and $C = 1$ in (12) and Theorem 1. However, in this case a more efficient inductive argument of [6] gives better estimates (4) and (5) above. If $q(u,z) \le f(u)$, then we may take $Q(s,t) = \int_s^t f(u) du$, $\eta = 0$ and $p = p^*$. In fact, if $q(u,z) = f(u)$, then $p_n(s,x,t,y) = Q(s,t)^n p(s,x,t,y)/n!$ and $\tilde{p}(s,x,t,y) = e^{Q(s,t)} p(s,x,t,y)$.

Less trivial applications of Theorem 1 require detailed study of p and q, and judicious choice of p^*. In particular, to estimate $\tilde{p}$ for given p, p^* and q, we wish to verify (16). This task may be facilitated by splitting the singularities of p and p^* in the integral of (16). Various versions of the 3G Theorem are used to this end, see [3], Bogdan and Jakubowski [5], and the "elliptic case" of Cranston, Fabes and Zhao [9], and Hansen [14]. For instance the transition density of the fractional Laplacian enjoys the following 3G

$$p(s,x,u,z) \wedge p(u,z,t,y) \le c\, p(s,x,t,y),$$

where $x, z, y \in \mathbb{R}^d$, $s < u < t$ ([5, Theorem 4]), and this yields

$$\begin{aligned} p(s,x,u,z)p(u,z,t,y) &= [p(s,x,u,z) \vee p(u,z,t,y)][p(s,x,u,z) \wedge p(u,z,t,y)] \\ &\le c\,[p(s,x,u,z) + p(u,z,t,y)] p(s,x,t,y). \end{aligned}$$

In this situation we can use $p^* = p$ in Theorem 1 to estimate $\tilde{p}$, provided

$$c \int_s^t \int_X p(s,x,u,z) q(u,z)\, dz du + c \int_s^t \int_X q(u,z) p(u,z,t,y)\, dz du \le \eta + Q(s,t)\,.$$

Noteworthy, 3G fails for Gaussian kernels, and for such kernels the methods of [6] fall short of optimal known results. This circumstance largely motivates

the present development. Below we show how to estimate quite general Schrödinger perturbations of Gaussian kernels by means of Theorem 1. The application depends on a 4G estimate, which is stated in (24) of Theorem 2 and partially substitutes for 3G. We note that Theorem 2 improves [26, (4.4)], since we give an optimal constant in (24). Explicit constants matter in applications, because we specifically require $\eta < 1/2$ in Theorem 1.

3 Estimates of the Gaussian kernel

As usual $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Let $0 < \alpha < \infty$, and

$$\begin{aligned} L(\alpha) &= \max_{\tau \geq \alpha \vee 1/\alpha} \left[\ln(1+\tau) - \frac{\tau - \alpha}{1+\tau} \ln(\alpha\tau) \right] \\ &= \max_{\tau \geq \alpha \vee 1/\alpha} \left[\ln\left(1 + \frac{1}{\tau}\right) - \ln\alpha + \frac{1+\alpha}{1+\tau} \ln(\alpha\tau) \right]. \end{aligned} \tag{19}$$

Clearly, $L(\alpha) < \infty$, and $\tau = \alpha \vee 1/\alpha$ yields $L(\alpha) \geq \ln(1 + \alpha \vee 1/\alpha)$. By an application of calculus, $L(\alpha) = \ln(1 + \alpha)$ if (and only if) $\alpha \geq e^{1/2}$. We let

$$f(\tau, x) = \ln\tau + x^2/\tau\,, \qquad \tau > 0,\ x \geq 0.$$

Lemma 5. *If $\alpha > 0$, $L = L(\alpha)$, $\xi, \eta \geq 0$, and $\tau > 0$, then*

$$f(1+\tau, \xi+\eta) \leq f(1,\xi) \vee f(\alpha\tau, \eta) + \frac{\eta^2}{\tau} + L. \tag{20}$$

If $L < L(\alpha)$, then the inequality fails for some $\xi, \eta \geq 0$ and $\tau > 0$.

Proof. We first prove the following implication:

$$\text{If} \quad \frac{\eta^2}{\alpha\tau} + \ln(\alpha\tau) \leq \xi^2, \quad \text{then} \quad \ln(1+\tau) \leq \frac{(\tau\xi - \eta)^2}{\tau(1+\tau)} + L\,. \tag{21}$$

To this end we consider two special cases:

Case 1. $\eta^2/(\alpha\tau) + \ln(\alpha\tau) \leq \xi^2$ and $\eta = \tau\xi$,

Case 2. $\eta^2/(\alpha\tau) + \ln(\alpha\tau) = \xi^2$ and $\eta < \tau\xi$.

Case 1 implies that $(\tau/\alpha - 1)\,\xi^2 + \ln(\alpha\tau) \leq 0$. This is possible only if $\tau \leq \alpha \vee 1/\alpha$, whence $\ln(1+\tau) \leq \ln(1+\alpha\vee 1/\alpha) \leq L(\alpha)$, which verifies (21).

In Case 2, if $\tau \leq \alpha \vee 1/\alpha$, then $\ln(1+\tau) \leq \ln(1+\alpha\vee 1/\alpha) \leq L(\alpha)$ again. For $\tau > \alpha \vee 1/\alpha$ we consider $\xi = \xi(\eta)$ as a function of η, and we have

$$\phi(\eta) := \ln(1+\tau) - \frac{(\tau\xi - \eta)^2}{\tau(1+\tau)} \leq L.$$

Indeed, we see that the condition $\eta < \tau\xi$ holds automatically since $\eta^2/\tau^2 \leq \eta^2/(\alpha\tau) = \xi^2 - \ln(\alpha\tau) < \xi^2$. Our assumption now reads $\xi^2 = \eta^2/(\alpha\tau) +$

$\ln(\alpha\tau)$, where $\xi, \eta \geq 0$ and $\tau > \alpha \vee 1/\alpha$. Thus, $\xi' = \eta/(\alpha\tau\xi)$. Note that $\phi(0) = \ln(1+\tau) - \tau\ln(\alpha\tau)/(1+\tau) \leq L$. Furthermore,

$$\phi'(\eta) = -2(1+\tau)^{-1}(\tau\xi - \eta)(\xi' - 1/\tau).$$

We have $\phi'(\eta) = 0$ only if $\xi' = 1/\tau$, or $\xi = \eta/\alpha$, and then $\eta^2/(\alpha\tau) + \ln(\alpha\tau) = \eta^2/\alpha^2$ and $\phi(\eta) = \ln(1+\tau) - (\tau-\alpha)\ln(\alpha\tau)/(1+\tau)$. This in fact shows that $L = L(\alpha)$ is sharp in (21), see (19). Furthermore, $\phi^{'}(\eta) \leq 0$ if $\xi' \geq 1/\tau$, or $(\tau/\alpha - 1)\,\eta^2/\alpha \geq \tau\ln(\alpha\tau)$, in particular if η is large. Thus, ϕ is decreasing for large η, which yields (21) in Case 2.

Consider general ξ, η and $\tau > 0$ in (21). If $\eta > \tau\xi$, then decreasing η to $\tau\xi$ strengthens (21), so eventually we are done by Case 1. If $\eta < \tau\xi$, then we increase η and strengthen the consequent of (21), getting under Case 1 or 2.

Putting (21) differently, $\ln(1+\tau) + (\xi+\eta)^2/(1+\tau) \leq \xi^2 + \eta^2/\tau + L$, provided $\eta^2/(\alpha\tau) + \ln(\alpha\tau) \leq \xi^2$. Therefore we have (20) under the assumption $f(\alpha\tau, \eta) \leq f(1, \xi)$, and the constant L cannot be improved. In particular, (20) holds if $f(1,\xi) = f(\alpha\tau, \eta)$. Decreasing ξ keeps (20) valid because $f(1+\tau, \xi+\eta)$ then decreases, too. □

For $d \in \mathbb{N}$, $a > 0$, $s < t$, and $x, y \in \mathbb{R}^d$, we consider the Gaussian kernel

$$g_a(s,x,t,y) = [4\pi(t-s)/a]^{-d/2} \exp\left\{ -|y-x|^2/[4(t-s)/a] \right\}. \tag{22}$$

Let $g = g_1$. Clearly, if $0 < a < b < \infty$, then for all $s < t$ and $x, y \in \mathbb{R}^d$,

$$g_b(s,x,t,y) \leq (b/a)^{d/2} g_a(s,x,t,y)\,. \tag{23}$$

We note that 3G fails for g_a, because

$$\frac{g_a(0,0,t,y) \wedge g_a(t,y,2t,2y)}{g_a(0,0,2t,2y)} = \frac{(4\pi t/a)^{-d/2} e^{-|y|^2/(4t/a)}}{(8\pi t/a)^{-d/2} e^{-|y|^2/(2t/a)}} = 2^{d/2} e^{a|y|^2/4t},$$

is not bounded in $y \in \mathbb{R}^d$. The next inequality (24) between four different instances of the Gaussian kernel substitutes for 3G, and so it is coined 4G. We note that [26, the proof of (4.4)] yields (24), too, although with a rough constant M (see also the first equality on p. 465 in [27] and the last equality on p. 15 in Friedman [12]). We also acknowledge a similar result (with rough constants) for the heat kernel of smooth bounded domains by Riahi [23, Lemma 3.1]. The optimality of the right-hand side of (24) is important in view of (16) and (17), and may be of independent interest. In fact, inspection of our calculations reveals that $b-a$ in g_{b-a} of (24) cannot be replaced by a bigger constant.

Theorem 2 (4G)**.** *Let $0 < a < b < \infty$ and $M = \left(\frac{b}{b-a}\right)^{d/2} \exp\left[\frac{d}{2}L(\frac{a}{b-a})\right]$. For all $s < u < t$ and $x, z, y \in \mathbb{R}^d$, we have*

$$g_b(s,x,u,z)g_a(u,z,t,y) \leq M[g_{b-a}(s,x,u,z) \vee g_a(u,z,t,y)]\, g_a(s,x,t,y), \tag{24}$$

which fails for some $s<u<t$, $x,z,y\in\mathbb{R}^d$, if $M<\left(\frac{b}{b-a}\right)^{d/2}\exp\left[\frac{d}{2}L(\frac{a}{b-a})\right]$. If $1/(1+e^{-1/2})\le a/b<1$, then $\left(\frac{b}{b-a}\right)^{d/2}\exp\left[\frac{d}{2}L(\frac{a}{b-a})\right]=(1-a/b)^{-d}$.

Proof. We have

$$\ln g_a(s,x,t,y)=-\frac{d}{2}\ln 4\pi-\frac{d}{2}\ln(t-s)+\frac{d}{2}\ln a-\frac{a|y-x|^2}{4(t-s)}. \tag{25}$$

Considering $\sqrt{2d}x$, $\sqrt{2d}y$ and $\sqrt{2d}z$ instead of x, y and z in (25), we see that (24) is equivalent to

$$\begin{aligned}&-\ln(u-s)+\ln b-\frac{b|z-x|^2}{u-s}-\ln(t-u)+\ln a-\frac{a|y-z|^2}{t-u}\\&\le\frac{2}{d}\ln M+\left[-\ln(u-s)+\ln(b-a)-\frac{(b-a)|z-x|^2}{u-s}\right]\vee\\&\left[-\ln(t-u)+\ln a-\frac{a|y-z|^2}{t-u}\right]-\ln(t-s)+\ln a-\frac{a|y-x|^2}{t-s}.\end{aligned}$$

We rewrite this using the identity $a+b-a\vee b=a\wedge b$, and we obtain

$$\begin{aligned}&\ln b-\ln(b-a)-\frac{a|z-x|^2}{u-s}+\left[-\ln(u-s)+\ln(b-a)-\frac{(b-a)|z-x|^2}{u-s}\right]\wedge\\&\left[-\ln(t-u)+\ln a-\frac{a|y-z|^2}{t-u}\right]\le\frac{2}{d}\ln M-\ln(t-s)+\ln a-\frac{a|y-x|^2}{t-s}.\end{aligned}$$

Adding $\ln(t-u)-\ln a$ to both sides, we have

$$\begin{aligned}\frac{a|y-x|^2}{t-s}+\ln\frac{t-s}{t-u}&\le\frac{2}{d}\ln M+\ln\frac{b-a}{b}\\&+\frac{a|y-z|^2}{t-u}\vee\left[\ln\frac{u-s}{t-u}+\ln\frac{a}{b-a}+\frac{(b-a)|z-x|^2}{u-s}\right]+\frac{a|z-x|^2}{u-s}.\end{aligned}$$

We denote $\alpha=a/(b-a)$, $\tau=(u-s)/(t-u)$, $\xi=|y-z|\sqrt{a/(t-u)}$ and $\eta=|z-x|\sqrt{a/(t-u)}$, and observe that $(t-s)/(t-u)=1+\tau$. Since $|y-x|\le|z-x|+|y-z|$, we see that (24) is equivalent to the following inequality (to hold for all $\tau>0$ and $\xi,\eta\ge 0$),

$$\frac{(\xi+\eta)^2}{1+\tau}+\ln(1+\tau)\le\frac{2}{d}M+\ln\frac{b-a}{b}+\xi^2\vee\left[\frac{\eta^2}{\alpha\tau}+\ln(\alpha\tau)\right]. \tag{26}$$

We may now use Lemma 5. In fact, the constant M in (24) is optimal if

$$\frac{2}{d}\ln M+\ln\frac{b-a}{b}=L(\alpha).$$

Considering $\alpha=a/(b-a)>e^{1/2}$, the last statement of the theorem follows from a comment after (19). $\square$

Remark 6. In applications we usually choose a (smaller than but) close to b, so to not loose much of Gaussian asymptotics, and in this case the optimality of mere $M = (1 - a/b)^{-d}$ comes as a nice feature of 4G.

4 Discussion and examples

In this section we discuss applications of Theorem 1 to fundamental solutions of second order parabolic differential operators. Namely, Theorem 1, aided by Theorem 2, allows for rather singular Schrödinger perturbations of such operators without dramatically changing the magnitude of their fundamental solutions. Most of the estimates given below are known, but our proofs are more synthetic and considerably shorter, and we have explicit constants in the estimates, which may be useful in homogenization problems. We also note that in the case of signed perturbations (not considered here) very precise lower bounds are obtained from Jensen's inequality for bridges [3], see also Remark 1 above. We begin with a discussion of Kato-type conditions (historical comments are given in Remark 8).

Let $d \geq 3$. A Borel function $U \colon \mathbb{R}^d \to \mathbb{R}$ is of Kato class, if

$$\lim_{\delta \to 0^+} \sup_{x \in \mathbb{R}^d} \int\limits_{|z-x|<\delta} \frac{|U(z)|}{|z-x|^{d-2}}\, dz = 0\,. \tag{27}$$

A typical example is $U(z) = |z|^{-2+\varepsilon}$, where $0 < \varepsilon \leq 2$. By Aizenman and Simon [1, Theorem 4.5] or Chung and Zhao [8, Theorem 3.6], (27) holds if and only if for every $c > 0$ the following condition is satisfied (see (22)),

$$\lim_{h \to 0^+} \sup_{s \in \mathbb{R},\, x \in \mathbb{R}^d} \int\limits_s^{s+h} \int\limits_{\mathbb{R}^d} g_c(s,x,u,z)|U(z)|\, dz du = 0\,. \tag{28}$$

In fact, there is a constant $c_1 = c_1(d,c)$ such that for all $h > 0$ and U,

$$\begin{aligned} c_1^{-1} \sup_{x \in \mathbb{R}^d} \int\limits_{|z-x|<\sqrt{h}} \frac{|U(z)|}{|z-x|^{d-2}}\, dz &\leq \sup_{s \in \mathbb{R},\, x \in \mathbb{R}^d} \int\limits_s^{s+h} \int\limits_{\mathbb{R}^d} g_c(s,x,u,z)|U(z)|\, dz du \\ &\leq c_1 \sup_{x \in \mathbb{R}^d} \int\limits_{|z-x|<\sqrt{h}} \frac{|U(z)|}{|z-x|^{d-2}}\, dz\,. \end{aligned} \tag{29}$$

The lower bound of (29) is given in [1, (4.5)] and [8, Lemma 3.5]. The upper bound can be proved as in [5, Lemma 11], but for the reader's convenience we give a simple, explicit and more flexible argument. Firstly, by changing variables $z = \sqrt{h}\tilde{z}$, $x = \sqrt{h}\tilde{x}$, $s = h\tilde{s}$, $\tilde{U}(z) = hU(\sqrt{h}z)$, we may and do

assume that $h = 1$. Then we consider the Newtonian kernel

$$K(x) = \int_0^\infty g(0,0,t,x)dt = |x|^{2-d}\Gamma(d/2-1)\pi^{-d/2}/4, \quad x \in \mathbb{R}^d,$$

and we let $k(x) = \int_0^1 g(0,0,t,x)dt$. Clearly, $\int_{\mathbb{R}^d} k(x)dx = 1$ and k is *radially decreasing*, i.e. $k(x_1) \geq k(x_2)$, if $|x_1| \leq |x_2|$. For $x \in \mathbb{R}^d$ and $r > 0$ we denote $B(x,r) = \{y \in \mathbb{R}^d : |y-x| < r\}$, and we consider $1_{B(0,1)}$, the indicator function of the unit ball. We observe the following auxiliary result.

Lemma 7. *If $f \geq 0$ is radially decreasing and f is constant on $B(0,1)$, then*

$$f * 1_{B(0,1)} \geq |B(0,1/2)| f.$$

Proof. We have $f * 1_{B(0,1)}(x) = \int_{B(x,1)} f(y)dy$. If $|x| < 1$, then

$$f * 1_{B(0,1)}(x) \geq f(0)|B(0,1) \cap B(x,1)| \geq f(0)|B(0,1/2)| = f(x)|B(0,1/2)|,$$

where $|B(0,1/2)|$ denotes the volume of $B(0,1/2)$. If $|x| \geq 1$, then

$$f * 1_{B(0,1)}(x) \geq f(x)|B(0,|x|) \cap B(x,1)| \geq f(x)|B(0,1/2)|.$$

$\square$

We let $f = k \wedge k(1,0,\ldots,0)$ and $\kappa(x) = K(x)1_{B(0,1)}$. By Lemma 7,

$$k \leq \kappa * (C_d f + \delta_0),$$

where $C_d = (|B(0,1/2)|\,\Gamma(d/2-1)\pi^{-d/2}/4)^{-1}$ and δ_0 denotes the Dirac measure at 0. The upper bound in (29) now follows for $c = 1$:

$$|U| * k \leq |U| * \kappa * (C_d f + \delta_0) \leq (1 + C_d) \sup |U| * \kappa.$$

The case of the upper bound in (29) for general $c > 0$ is similar.

We now pass to *parabolic Kato condition*. For $c > 0$, $h > 0$ and $V : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ we denote

$$N_h^c(V) = \sup_{s,x} \int_s^{s+h} \int_{\mathbb{R}^d} g_c(s,x,u,z)|V(u,z)|dzdu + \sup_{t,y} \int_{t-h}^{t} \int_{\mathbb{R}^d} g_c(u,z,t,y)|V(u,z)|dzdu.$$

We say that V is of the *parabolic* Kato class if $\lim_{h\to 0^+} N_h^c(V) = 0$ for every $c > 0$. Considering $V(s,x) = U(x)$ for $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, we may regard the parabolic Kato class as wider than the (time-independent) Kato class. We note that $N_h^c(V)$ is non-decreasing in h. Since g_c is a probability transition density, $N_{h_1+h_2}^c(V) \leq N_{h_1}^c(V) + N_{h_2}^c(V)$, hence

$$N_{t-s}^c(V) \leq N_h^c(V) + N_h^c(V)(t-s)/h, \quad h > 0. \tag{30}$$

To focus on nonnegative Schrödinger perturbations (in this connection see Remark 1), we consider, as before, Borel function $q \geq 0$ on space-time. If $0 < a < b < \infty$, then by 4G Theorem, there is a constant M' depending only on d and b/a, such that for all $s < t$ and $x, y \in \mathbb{R}^d$,

$$\int_s^t \int_{\mathbb{R}^d} g_b(s,x,u,z) q(u,z) g_a(u,z,t,y)\, dz du \leq M' N^c_{t-s}(q)\; g_a(s,x,t,y)\,, \quad (31)$$

where $c = (b-a) \wedge a$. Indeed, by (23) we can take $M' = \left(\frac{b-a}{a} \vee \frac{a}{b-a}\right)^{d/2} M$, where M is the constant from 4G Theorem. Therefore if $q \geq 0$ belongs to the parabolic Kato class and $0 < a < b < \infty$, then by (30) and (31) we have

$$q \in \mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q), \quad (32)$$

with $Q(s,t) = \beta(t-s)$ and $\beta = \eta/h$, provided $h > 0$ and $\eta > 0$ are such that $N_h^{(b-a)\wedge a}(q) \leq \eta/M'$. Clearly, the condition allows for arbitrarily small $\eta > 0$.

Remark 8. The (time-independent) Kato class was first used to perturb the Laplace operator by Aizema and Simon [1], and was characterized as smallness with respect to the Laplacian on $L^1(\mathbb{R}^d)$. The parabolic Kato class was proposed for the Gaussian kernel by Zhao in [26]. It was then generalized and used by Liskevich and Semenov [19], Liskevich, Vogt and Voigt [20] and Gulisashvili and van Casteren [13]. The condition is related to Miyadera perturbations of the semigroup of the Laplacian on $L^1(\mathbb{R}^d)$, see Schnaubelt and Voigt [24]. Time-independent Kato class is wider than $L^p(\mathbb{R}^d)$ if $p > d/2$ [1], [8, Chapter 3., Example 2]. Nevertheless, the latter space is quite natural for perturbing Gaussian kernels, see Aronson [2], Dziubański and Zienkiewicz [11], [27, Remark 1.1(b)]. Another Kato-type condition was introduced by Zhao in [27] to obtain strict comparability of g and $\tilde{g}$. As noted in [27, Remark 1.1(c)], the condition may be formulated in terms of *Brownian bridges*. This point of view was later developed in [3] (under the name of the relative Kato condition) and elaborated in [6] to

$$\int_s^t \int_X \frac{p(s,x,u,z)p(u,z,t,y)}{p(s,x,t,y)} q(u,z) dz du \leq [\eta + Q(s,t)], \quad (33)$$

where $s < t$, $x, y \in X$, $\eta < \infty$, and $0 \leq Q(s,u) + Q(u,t) \leq Q(s,t)$, cf. (1). Condition (33) indicates why we mention bridges here (see [3] for details). The Kato condition for bridges gives better upper bounds and seems more intrinsic to Schrödinger perturbations than the parabolic Kato condition, but the former may be cumbersome to verify in concrete situations. For the classical Gaussian kernel, (33) is stronger than the corresponding parabolic

Kato condition with a fixed c (see [3, Lemma 9] for a more general result), and it is rather difficult to explicitly characterize [27, Remark 1.2(a,b)]. This is due to the relatively large values of the integrand in (33) for (u,z) on the interval connecting (s,x) and (t,y). If p satisfies 3G inequality, $p(s,x,t,y) = p(s,y,t,x)$ and p is a probability transition density, then the parabolic Kato class and the Kato class for bridges are equivalent. This is the case for the transition density of the fractional Laplacian $\Delta^{\alpha/2}$ with $\alpha \in (0,2)$ [3, Corollary 11], and the proof of this fact is similar to our application of 4G in (31). We emphasize that each transition density p determines its specific Kato classes (either parabolic and for bridges), and detailed analysis is required to manage particularly singular q.

Let $d \geq 3$, $z_1 \in \mathbb{R}^d$, $|z_1| = 1$, and $B(nz_1, 1/n) \subset \mathbb{R}^d$ be the ball with radius $1/n$ and center nz_1, $n = 1, 2, \ldots$. We define

$$U(z) = \sum_{n=2}^{\infty} n|z - nz_1|^{-1} \mathbf{1}_{B(nz_1,1/n)}(z), \quad z \in \mathbb{R}^d.$$

If $\delta > 0$ and $n \geq 1/\delta$, then

$$\sup_{x\in\mathbb{R}^d} \int\limits_{|z-x|<\delta} \frac{U(z)}{|z-x|^{d-2}}\, dz \geq \int\limits_{B(nz_1,1/n)} n|z - nz_1|^{-d+1}\, dz = \int\limits_{B(0,1)} |z|^{-d+1}\, dz\,.$$

Therefore εU does not belong to the parabolic Kato class for any $\varepsilon > 0$. On the other hand, by (29) we have,

$$N_1^c(U) \leq 2c_1 \sup_{x\in\mathbb{R}^d} \int\limits_{|z-x|^2<1} \frac{U(z)}{|z-x|^{d-2}}\, dz < \infty\,,$$

hence $\varepsilon U \in \mathcal{N}(g_b, g_a, (b/a)^{d/2}, \eta, Q)$ with *linear* Q and $\eta < 1/2$, provided ε is sufficiently small, cf. (31), (30). Since our constants are quite explicit, so are the resulting upper bounds for $\tilde{g}_b$. To close the discussion of examples of q manageable by our methods, we let $q(u,z) := U(z) + f(z,u)$, where $f \geq 0$, U is as above, and $\sup_{z\in\mathbb{R}^d} f(u,z)$ is finite and increases to infinity as $u \to \infty$. Such q requires $\eta > 0$ to control U, and a *superlinear* Q to majorize $\int_s^t \sup_{z\in\mathbb{R}^d} f(u,z)\, du$.

We next focus on Gaussian estimates and consider a measurable transition density p on $X = \mathbb{R}^d$ with the Borel sets $\mathcal{M} = \mathcal{B}_{\mathbb{R}^d}$ and the Lebesgue measure $m(dz) = dz$. Assume that $b > 0$, $\Lambda \geq 1$ and $\lambda \in \mathbb{R}$ exist such that for $s < t$ we have

$$p(s,x,t,y) \leq \Lambda e^{\lambda(t-s)} g_b(s,x,t,y), \quad x, y \in \mathbb{R}^d\,. \tag{34}$$

Let $0 < a < b$ and $p^*(s,x,t,y) = e^{\lambda(t-s)} g_a(s,x,t,y)$. In view of (23) we put $C = \Lambda(b/a)^{d/2}$. If $q \in \mathcal{N}(g_b, g_a, \frac{C}{\Lambda}, \frac{\eta}{\Lambda}, \frac{Q}{\Lambda})$, which is a condition on η and Q,

then $q \in \mathcal{N}(p, p^*, C, \eta, Q)$, and by Theorem 1, for all $s < t$ and $x, y \in \mathbb{R}^d$,

$$\tilde{p}(s,x,t,y) \le \left(\frac{C}{1-2\eta}\right)^{1+\frac{Q(s,t)}{\eta}} e^{\lambda(t-s)} g_a(s,x,t,y) \quad \text{if } 0 < \eta < 1/2\,, \tag{35}$$

$$\tilde{p}(s,x,t,y) \le (2C)^{1+2Q(s,t)} e^{\lambda(t-s)} g_a(s,x,t,y), \quad \text{if } \eta = 0\,. \tag{36}$$

In particular, such $\eta > 0$ may be chosen arbitrarily small, with a linear Q, if q is in the Kato class or the parabolic Kato class (see the discussion above). This is our general setup for Gaussian estimates.

Example 1. If $p = g_b$ and $p^* = g_a$, then we take $\Lambda = 1$, $C = (b/a)^{d/2}$ and $\lambda = 0$ in (34) and, consequently, in (35,36). For clarity, a, the coefficient in the exponent of the Gaussian majorant, may be arbitrarily close to b, at the expense of the factor before the majorant in (35,36), and we require that $q \in \mathcal{N}(g_b, g_a, C, \eta, Q)$ with $\eta \in [0, 1/2)$, which is satisfied, e.g., for q in the the Kato class. We thus recover the best results known in this setting [26, proof of Theorem A] with explicit control of constants.
To relate our results to second order differential operators, we let $C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ denote the smooth compactly supported functions on space-time, and we recall that for all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$,

$$\int_s^\infty \int_{\mathbb{R}^d} p(s,x,u,z) \left[\frac{\partial \phi(u,z)}{\partial u} + \frac{1}{b}\Delta\phi(u,z)\right] dz du = -\phi(s,x)\,. \tag{37}$$

The identity may, for instance, be obtained from integration by parts or by using Fourier transform in the space variable. By a general result, [6, Lemma 4], the perturbed transition density $\tilde{p}$ corresponds to the Schrödinger-type operator $\frac{1}{b}\Delta + q$ in the same way,

$$\int_s^\infty \int_{\mathbb{R}^d} \tilde{p}(s,x,u,z) \left[\frac{\partial \phi(u,z)}{\partial u} + \frac{1}{b}\Delta\phi(u,z) + q(u,z)\phi(u,z)\right] dz du = -\phi(s,x)\,,$$

provided $q \in \mathcal{N}(g_b, g_a, C, \eta, Q)$ with $\eta \in [0, 1/2)$, as above.

Example 2. Let p be the transition density of the one-dimensional Brownian motion with constant unit drift,

$$p(s,x,t,y) = g_1(s, x-s, t, y-t)\,, \qquad s < t,\ x, y \in \mathbb{R}\,.$$

There are no constants c_1, c_2 such that $p(s,x,t,y) \le c_1 g_{c_2}(s,x,t,y)$ for all $s < t$ and $x, y \in \mathbb{R}$ (cf. Zhang [25, Remark 1.3]). On the other hand, for each $b \in (0,1)$ we have

$$p(s,x,t,y) \le b^{-1/2} e^{\frac{b}{4(1-b)}(t-s)} g_b(s,x,t,y)\,, \qquad s < t,\ x, y \in \mathbb{R}\,.$$

This shows why we may need $\lambda \ne 0$ in (34), see also Norris [22].

Example 3. Let $f : \mathbb{R} \times \mathbb{R}^d \ni (s, x) \mapsto \mathbb{R}$, and

$$Lf = \sum_{i,j=1}^{n} a_{ij}(s,x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(s,x) \frac{\partial f}{\partial x_i},$$

be a uniformly elliptic operator, with bounded uniformly Hölder continuous coefficients b_i and $a_{ij} = a_{ji}$, see Dynkin [10, Chapter 2, 1.1.A and 1.1.B] for detailed definitions, and [12, Chapter 1] for wider perspective. Consider the fundamental solution $p(s, x, t, y)$ in the sense of [10, Theorem 1.1] for the following differential parabolic operator

$$\frac{\partial f}{\partial s} + Lf. \tag{38}$$

By the results of [10, Chapter 2], in particular Theorem 1.1, 1.3.1 and 1.3.3, p satisfies our assumptions, including (34) with $-\infty < t_1 \le s < t \le t_2 < \infty$. Therefore the Schrödinger perturbation $\tilde{p}$ of p satisfies (35) or (36) in the (finite) time horizon $[t_1, t_2]$, if $q \in \mathcal{N}(g_b, g_a, \frac{C}{\Lambda}, \frac{\eta}{\Lambda}, \frac{Q}{\Lambda})$ and $\eta \in [0, 1/2)$, as explained before Example 1 and after the proof of Theorem 1. We thus recover recent results of [20, Theorem 3.10 and 3.12] (see also Remark 9 below). We now explain how p and $\tilde{p}$ are related to parabolic operators. For all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$, we have

$$\int_s^\infty \int_{\mathbb{R}^d} p(s, x, u, z) \left[\frac{\partial \phi(u, z)}{\partial u} + L\phi(u, z) \right] dz du = -\phi(s, x). \tag{39}$$

In fact, the identity holds if the function $\phi(s, x)$ is bounded, supported in a finite time interval and uniformly Hölder continuous in x, and if the same is true for its first derivative in time and all its derivatives up to the second order in space. Indeed, if we let

$$h(s, x) = \phi(s, x) + \int_s^\infty \int_{\mathbb{R}^d} p(s, x, u, z) \left[\frac{\partial \phi(u, z)}{\partial u} + L\phi(u, z) \right] dz du,$$

then by [10, Chapter 2, 1.3.3] we have $h \equiv 0$, which verifies (39). By (39) and [6, Lemma 4], the perturbed transition density $\tilde{p}$ corresponds to the Schrödinger-type operator $L + q$ in a similar way: for all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$,

$$\int_s^\infty \int_{\mathbb{R}^d} \tilde{p}(s, x, u, z) \left[\frac{\partial \phi(u, z)}{\partial u} + L\phi(u, z) + q(u, z)\phi(u, z) \right] dz du = -\phi(s, x).$$

Remark 9. In this paper by a fundamental solution we mean the negative of an integral inverse of a given operator acting on space-time (other authors

also use the terms heat kernel and Green function). More specifically, our p and $\tilde{p}$ may be considered post-inverses of the respective differential operators acting on $C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$, cf. Example 1 above and [27], [23, p. 13]. In the literature on partial differential equations it is also common to consider the pre-inverses, which necessitates that the applications of p, $\tilde{p}$ to test functions be sufficiently differentiable [12, Theorem I.5.9], [10, (1.12)]. Still differently, if the operator L is in the divergence form, a common notion is that of the weak fundamental solution, related to integration by parts, see Aronson [2, (2), (8), Section 6 and 7], Cho, Kim and Park [7] and Liskevich and Semenov [19]. It is also customary to study the operator $\partial f/\partial s - Lf$, which is related to (38) by time reversal $s \mapsto -s$ [10, 13], [2, (7.3)]. The setting of (38) and (39) is most appropriate from the probabilistic point of view: it agrees with time precedence and notation for (measurable) transition densities of Markov processes, which may be conveniently considered as integral operators on space-time.

Remark 10. As we already mentioned, Zhang [27] gives sharp estimates for perturbations of $p = g$. Sharp Gaussian estimates (corresponding to $p^* = p$) are generally not available by our methods if parabolic Kato condition and 4G should be used to estimate p_1. Accordingly, [27] assumes an integral condition related to the Brownian bridge, to bound p_1 by p. As we explained in Remark 8, the Kato condition for bridges is more restrictive than the parabolic Kato condition (a straightforward general approach using bridges is given in [6]).

We now comment on the uniqueness of $\tilde{p}$. Trivially, $\tilde{p}$ is unique because it is given by (2), rather than implicitly. However, in the literature of the subject a departure point for defining $\tilde{p}$ is usually one of the following Duhamel's (perturbation/resolvent) formulas,

$$\tilde{p}(s,x,t,y) = p(s,x,t,y) + \int_s^t \int_{\mathbb{R}^d} p(s,x,u,z)q(u,z)\tilde{p}(u,z,t,y)dudz, \tag{40}$$

$$\tilde{p}(s,x,t,y) = p(s,x,t,y) + \int_s^t \int_{\mathbb{R}^d} \tilde{p}(s,x,u,z)q(u,z)p(u,z,t,y)dudz. \tag{41}$$

In short, $\tilde{p} = p + pq\tilde{p}$ *or* $\tilde{p} = p + \tilde{p}qp$, depending on whether we consider p and $\tilde{p}$ as pre- *or* post-inverses, respectively, of the corresponding differential operators (see [6, 4] for notation related to integral kernels). Clearly, (2) yields (40) and (41). Conversely, iterating (40) or (41) we get (2), and uniqueness, provided $(pq)^n\tilde{p}$ or $\tilde{p}(qp)^n$ converge to zero as $n \to \infty$. So is the case with $(pq)^n\tilde{p}$ under the assumptions of Theorem 1, *if* $\tilde{p}$ is locally in time majorized by a constant multiple of p^*, because then $(pq)^n p^* \to 0$. We refer to [3, Theorem 2] and [20, Theorem 1.16] for further discussion of the

perturbation formula and uniqueness. We also note that in the setting of *bridges*, there is a natural probabilistic Feynman-Kac type formula for $\tilde{p}$ [3, Section 6], which readily yields (2) and uniqueness.

Here is a general argument leading to (41). We consider $-p$ and $-\tilde{p}$ as integral operators on space-time and post-inverses of operators L and $L+q$, respectively, which in turn act on the same given set of functions. We have $\tilde{p}(L\phi + q\phi) = -\phi = pL\phi$, hence $\tilde{p}\psi = p\psi + \tilde{p}qp\psi$, where $\psi = L\phi$. If the range of L uniquely determines measures, then we obtain $\tilde{p} = p + \tilde{p}qp$ as integral kernels (not necessarily pointwise). This is the case, e.g., in the context of Example 3. We finally note that some majorization of $\tilde{p}$ is needed for uniqueness. For instance, both $p = g_b$ and $p(s,x,u,z) = g_b(s,x,u,z) + 2dbu + |z|^2$ satisfy (37).

5 Miscellanea

Earlier work by Jakubowski [15] and coauthors [6] in slightly different settings does not require continuity assumptions on Q. Namely we call $Q\colon \mathbb{R}\times\mathbb{R} \to [0,\infty)$ *superadditive*, if

$$Q(s,u) + Q(u,t) \le Q(s,t)\,, \qquad \text{for} \quad s<u<t\,. \tag{42}$$

For convenience we define $Q(s,t) = 0$ if $s \ge t$. We see that $t \mapsto Q(s,t)$ is non-decreasing, $s \mapsto Q(s,t)$ is non-increasing, and the following limit exists,

$$Q^-(s,t) = \lim_{h\to 0^+} Q(s+h, t-h)\,. \tag{43}$$

For instance, if μ is a Radon measure on $\mathbb{R}$ and $Q(s,t) = \mu(\{u \in \mathbb{R} : s \le u < t\})$, then $Q^-(s,t) = \mu(\{u \in \mathbb{R} : s < u < t\})$.

Clearly, $0 \le Q^-(s,t) \le Q(s,t)$ and $Q^-(s,u) + Q^-(u,t) \le Q^-(s,t)$ if $s \le u \le t$. We have $Q^{--} = Q^-$. In fact, $Q^-(u,v) \to Q^-(s,t)$ as $u \to s^+$, $v \to t^-$, because if $0 < h < u-s < k$ and $h < t-v < k$, then $Q(s+k,t-k) \le Q^-(u,v) \le Q(s+h,t-h)$. In particular, $s \mapsto Q^-(s,t)$ is right-continuous and $t \mapsto Q^-(s,t)$ is left-continuous. We thus obtain the following result.

Corollary 11. *$Q^-(s,t)$ is regular superadditive.*

We note that continuous superadditive functions are used in the theory of rough paths by Lyons [21]. There are further similarities due to the role of iterated integrals in time here and in [21], and many differences related to the fact that we require absolute integrability (but see [16]) and also integrate/average in space (see (16) and (7)). We also note that methods similar to ours allow to handle gradient perturbations, which will be discussed in a forthcoming paper (see also [18, 16]). The next result shows that if Q is (plain) superadditive, then the factor $\eta + Q(s,t)$ in (16) may be replaced

with $C\eta + CQ^-(s,t)$, where C comes from (12), and CQ^- is regular superadditive. Thus, we may assure *regular* superadditivity at the expense of increasing η and Q.

Lemma 12. *Assume that p and p^* are transition densities, function $q \geq 0$ is measurable on space-time, $\eta \geq 0$, $C \geq 1$, Q is superadditive, and (12) and (16) hold. Then for all $s < t < \infty$ and $x, y \in X$, we have*

$$\int_s^t \int_X p(s,x,u,z)q(u,z)p^*(u,z,t,y)\, dz\, du \leq C\big[\eta + Q^-(s,t)\big]p^*(s,x,t,y)\,.$$

Proof. For $x, y \in X$, $s < t$, $h > 0$, by Chapman-Kolmogorov, (16) and (12),

$$\begin{aligned}
&\int_{s+h}^{t-h} \int_X p(s,x,u,z)q(u,z)p^*(u,z,t,y)\, dzdu = \int_X \int_X p(s,x,s+h,v) \\
&\int_{s+h}^{t-h} \int_X p(s+h,v,u,z)q(u,z)p^*(u,z,t-h,w)dzdu\, p^*(t-h,w,t,y)\, dwdv \\
&\leq \big[\eta + Q(s+h,t-h)\big] \int_X p(s,x,s+h,w)p^*(s+h,w,t,y)\, dw \qquad (44) \\
&\leq C\big[\eta + Q(s+h,t-h)\big]p^*(s,x,t,y)\,.
\end{aligned}$$

We then let $h \to 0$, and use (43) and the monotone convergence theorem. □

If p^* is a time-changed p, then we can do even better.

Lemma 13. *Under the assumptions of Lemma 12 we have*

$$\int_s^t \int_X p(s,x,u,z)q(u,z)p^*(u,z,t,y)\, dz\, du \leq \big[\eta + Q^-(s,t)\big]p^*(s,x,t,y)\,,$$

if $(s,\infty) \ni t \mapsto p(s,x,t,y)$ is continuous, p is time-homogeneous: $p(s,x,t,y) = p(s+r,x,t+r,y)$ for $r \in \mathbb{R}$, and $p^*(s,x,t,y) = p(as,x,at,y)$ for some $a > 0$.

Proof. Picking up (44), for $s < t$, $x, y \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\limsup_{h\to 0^+} \int_{\mathbb{R}^d} p(s,x,t-h,w)p^*(t-h,w,t,y)\, dw \\
&= \limsup_{h\to 0^+} \int_{\mathbb{R}^d} p(s,x,t-h,w)p(t-h,w,t+ah-h,y)\, dw \\
&= \limsup_{h\to 0^+} p(s,x,t+ah-h,y) = p(s,x,t,y)\,.
\end{aligned}$$

□

Lemma 13 applies to the Gaussian density, if $p = g_b$ and $p^* = g_a$, where $0 < a < b$. Indeed, $g_b(s, x, t, y) = g_a(as/b, x, at/b, y)$, see also (23).
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